Functional RG approach to nonequilibrium transport through mesoscopic systems

DPG spring meeting – TUT 3.2

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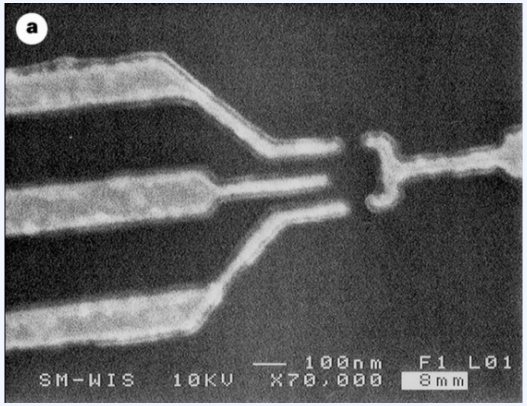
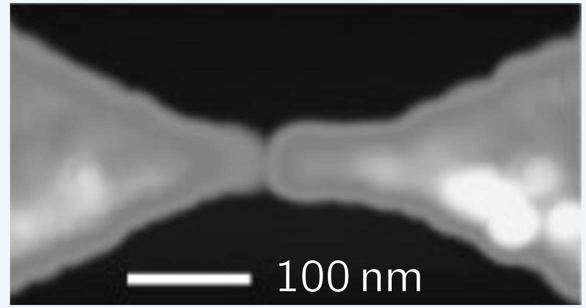
Motivation: mesoscopic transport

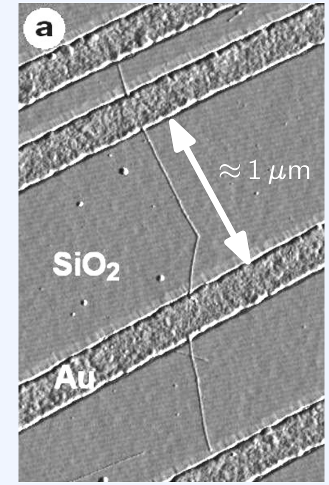
Mesoscopic samples



Goldhaber-Gordon et al., Nature '98

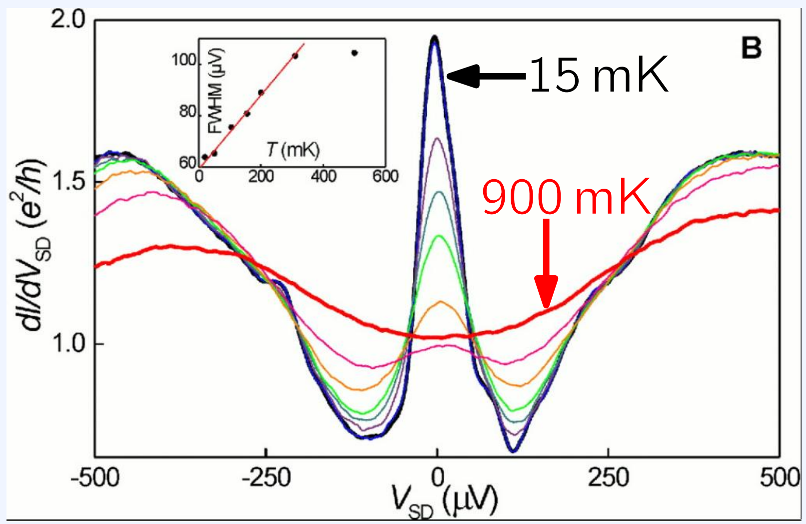


Park et al., Nature '02



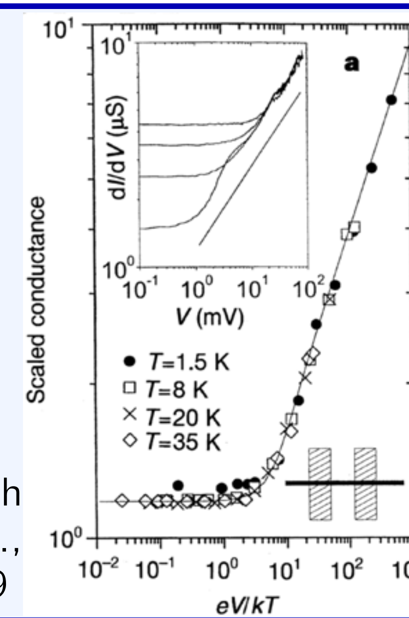
Yao et al., Nature '99

Measurements in nonequilibrium



van der Wiel et al., Science '00

Bockrath et al., Nature '99



coherent interacting open quantum system in nonequilibrium
$\Rightarrow$ Keldysh formalism

Pert. theory $\rightarrow$ divergencies,
mean field $\rightarrow$ artifacts
$\Rightarrow$ Functional (or other) RG

Overview

1. Keldysh formalism and generating functionals
2. Functional RG
3. Cut-offs for the Keldysh FRG
4. Examples:
   - Wire with strong contact barriers in steady state noneq.
   - transient behaviour of the IRLM after attaching leads

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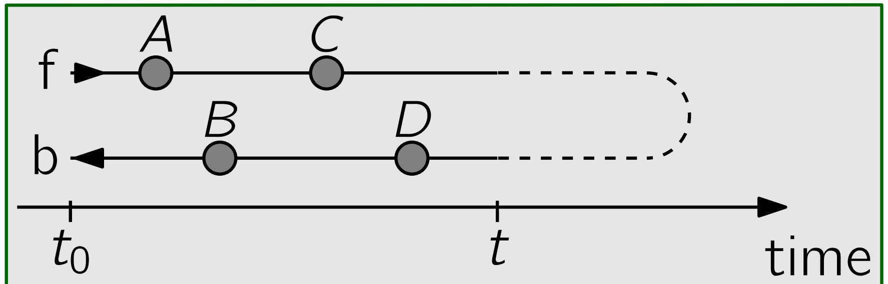
Time loop contour

System preparation: $\rho(t_0) = \rho_0 = \exp\left(-\sum_{s's} F_{s's} a^\dagger_{s'} a_s\right)/Z$ (uncorrelated)



$\mathcal{T}_c\, ABCD = BDCA$

- single-particle eigenstates of F are independently occupied
- if ρ_0 correlated: additional contour branch with correlation vertices

Time evolution:

$$\begin{aligned}
\langle X\rangle(t) &= \mathrm{Tr}\; U(t_0,t)\, X(t)\, U(t,t_0)\, \rho_0 \\
&= \mathrm{Tr}\left(\widetilde{\mathcal{T}} \exp\left[i\int_{t_0}^{t} H(\tau) d\tau\right]\right) X(t) \left(\mathcal{T} \exp\left[-i\int_{t_0}^{t} H(\tau) d\tau\right]\right) \rho_0 \\
&= \mathrm{Tr}\; \mathcal{T}_c\; \exp\left[-i\int_{t_0}^{\infty} [H_f(\tau) - H_b(\tau)] d\tau\right] X_f(t)\, \rho_0
\end{aligned}$$

Green functions and generating functional

Green function

$$G(1,\ldots,n|1',\ldots,n') \qquad \Big[1 = \big(\text{branch } j_1, \text{time } t_1, \text{state } s_1\big)\Big]$$

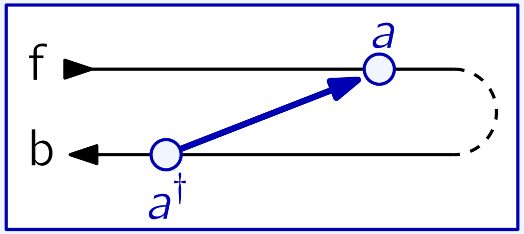
$$= (-i)^n \ \mathrm{Tr}\ \mathcal{T}_\mathrm{c}\ a_1^H \ldots a_n^H\ a_{n'}^{\dagger H} \ldots a_{1'}^{\dagger H}\ \rho_0$$

$$= (-i)^n \ \mathrm{Tr}\ \mathcal{T}_\mathrm{c}\ \exp\left[-i\int_{t_0}^{\infty}(H_\mathrm{f} - H_\mathrm{b})d\tau\right]\ a_1 \ldots a_n\ a_{n'}^{\dagger} \ldots a_{1'}^{\dagger}\ \rho_0$$

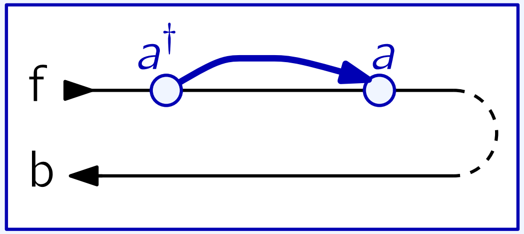
Single-particle Green function:

$$\begin{pmatrix} G_{\mathrm{f|f}} & G_{\mathrm{f|b}} \\ G_{\mathrm{b|f}} & G_{\mathrm{b|b}} \end{pmatrix} = \begin{pmatrix} G^{\mathrm{c}} & G^{<} \\ G^{>} & G^{\tilde{\mathrm{c}}} \end{pmatrix}$$

$$G^{<} = -i\langle \mathcal{T}_\mathrm{c}\, a_\mathrm{f} a_\mathrm{b}^{\dagger}\rangle = +i\langle a^\dagger a\rangle$$



$$G^{\mathrm{c}}(t|t') = \begin{cases} G^{<}, & t \le t', \\ G^{>}, & t > t', \end{cases}$$



$$G^{<} + G^{>} = G^{\mathrm{c}} + G^{\tilde{\mathrm{c}}}$$

Green functions and generating functional

Green function

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$$= (-i)^n \;\mathrm{Tr}\; \mathcal{T}_c \; a_1^H \ldots a_n^H \; a_{n'}^{\dagger H} \ldots a_{1'}^{\dagger H} \, \rho_0$$

$$= (-i)^n \;\mathrm{Tr}\; \mathcal{T}_c \; \exp\left[-i \int_{t_0}^{\infty} (H_f - H_b) d\tau\right] a_1 \ldots a_n \; a_{n'}^{\dagger} \ldots a_{1'}^{\dagger} \, \rho_0$$

Generating functional

$$H_{J,\bar{J}}(t) \;=\; H(t) + \sum_s \left[\bar{J}_s(t) a_s + a_s^{\dagger} J_s(t)\right] \qquad \text{Hamilt. with part. sources}$$

$$U_{J,\bar{J}}(t,t') \;=\; \text{time evolution under } H_{J,\bar{J}}$$

$$\mathcal{G}[J,\bar{J}] \;=\; \mathrm{Tr}\; U_{-J^b,-\bar{J}^b}(t_0,\infty)\; U_{J^f,\bar{J}^f}(\infty,t_0)\; \rho_0$$

$$G(1\ldots|\ldots n') = \; (-i)^n \, \frac{\delta^{2n}}{\delta\bar{J}_1 \ldots \delta\bar{J}_n \delta J_{n'} \ldots \delta J_{1'}} \, \mathcal{G}[J,\bar{J}] \Big|_{J=0=\bar{J}}$$

Green functions and generating functional

Functional integral representation:

$$\mathcal{G}[J,\bar{J}] \;=\; \frac{1}{Z}\int \mathcal{D}[\phi,\bar{\phi}]\, e^{\,iS[\phi,\bar{\phi}] \,-\, i\bar{J}\cdot\phi \,-\, i\bar{\phi}\cdot J}$$

$$S[\phi,\bar{\phi}] \;=\; Q_{1'1}\,\bar{\phi}_{1'}\phi_1 \;-\; \frac{1}{4}\bar{v}_{1'2'12}\bar{\phi}_{1'}\bar{\phi}_{2'}\phi_2\phi_1 \;=\; \text{action},$$

H_0, ρ_0 (→ $Q_{1'1}$) H_{int} (→ $\bar{v}_{1'2'12}$)

noninteracting: $\mathcal{G}_0[J,\bar{J}] \;=\; \exp\left[-i\bar{J}\cdot Q^{-1}J\right] \quad\Rightarrow\quad G_0 = Q^{-1}$

Generating functional

$$H_{J,\bar{J}}(t) \;=\; H(t) + \sum_s \left[\bar{J}_s(t)a_s + a_s^{\dagger}J_s(t)\right] \qquad \text{Hamilt. with part. sources}$$

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Mixing contour branches: The Keldysh rotation

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_\mathrm{f} - a_\mathrm{b} \\ a_\mathrm{f} + a_\mathrm{b} \end{pmatrix}$$

$$a_k = K_{kj} a_j, \quad k = 1, 2 = \text{Keldysh index}$$

$$a_k^\dagger = a_j^\dagger K_{jk}^\dagger = a_j^\dagger K_{jk}^{-1}$$

$$G = (-i)^n \,\mathrm{Tr}\, \mathcal{T}_\mathrm{c}\, a \ldots a a^\dagger \ldots a^\dagger \rho_0$$

$$G \to K \ldots K\, G\, K^{-1} \ldots K^{-1}$$

$$\bar{\nu} \to K K\, \bar{\nu}\, K^{-1} K^{-1}$$

$\to$ all diagram rules unchanged.

$$\frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} G^\mathrm{c} & G^< \\ G^> & G^{\tilde{\mathrm{c}}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} G_{1|1} & G_{1|2} \\ G_{2|1} & G_{2|2} \end{pmatrix} = \begin{pmatrix} 0 & G^\mathrm{A} \\ G^\mathrm{R} & G^\mathrm{K} \end{pmatrix}$$

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Free propagator:

$$G_0^\mathrm{R}(t|t') = -i\,\Theta(t-t')\,U_0(t,t')$$

$$G_0^\mathrm{K}(t|t') = \left(G_0^\mathrm{R} \Sigma_\mathrm{ini}^\mathrm{K} G_0^\mathrm{A}\right)(t|t')$$

$$\Sigma_\mathrm{ini}^\mathrm{K}(t'|t) = -i\left[1 - 2\langle a^\dagger a\rangle(t_0)\right]\delta(t'-t_0)\delta(t-t_0)$$

describes initial occupancy

In a stationary state:

$$G_0^\mathrm{R}(\omega) = \frac{1}{\omega - \epsilon + i\eta}$$

$$G_0^\mathrm{K}(\omega) = -2\pi i\left[1 - 2\langle a^\dagger a\rangle_0\right]\delta(\omega-\epsilon)$$

$$G^\mathrm{R} \to \begin{cases} \text{response,} \\ \text{spectrum} \end{cases}$$

$$G^\mathrm{K} \to \begin{cases} \text{occupancies,} \\ \text{fluctuations} \end{cases}$$

Mixing contour branches: The Keldysh rotation

equation for retarded component decoupled

Dyson's equation

$$G^{\mathrm{R}} = \left(G_0^{\mathrm{R}\,-1} - \Sigma^{\mathrm{R}}\right)^{-1}$$

$$G^{\mathrm{K}} = G^{\mathrm{R}}(\Sigma^{\mathrm{K}}_{\mathrm{ini}} + \Sigma^{\mathrm{K}})G^{\mathrm{A}}$$

initial occupancy

reshuffled occupancy due to interaction

Free propagator:

$$G_0^{\mathrm{R}}(t|t') = -i\,\Theta(t-t')\,U_0(t,t')$$

$$G_0^{\mathrm{K}}(t|t') = \left(G_0^{\mathrm{R}}\Sigma^{\mathrm{K}}_{\mathrm{ini}}G_0^{\mathrm{A}}\right)(t|t')$$

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  $\rightarrow$ two contour branches
- Green functions
  $=$ expansion coeff. of gen. functional
- $G^{\mathrm{R}} \rightarrow$ spectrum, $G^{\mathrm{K}} \rightarrow$ fluctuations

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Keldysh formalism: Haug, Jauho, Springer '96
Gen. func.: Negele, Orland, Addison-Wesley '88

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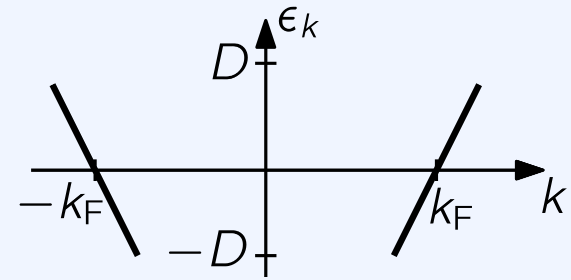
Logarithmic divergence in PT

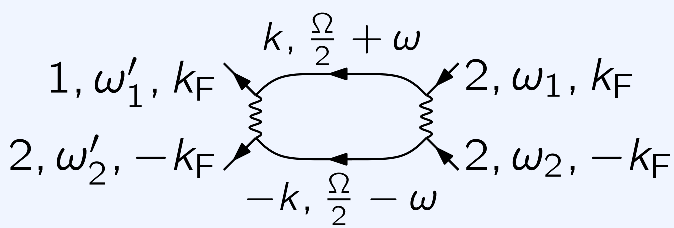
Example system

1D spinless Fermions, T=0,

linearized dispersion $\epsilon_k = v_\mathrm{F}(|k| - k_\mathrm{F})$,

finite bandwidth $2D$





$$\sim \int_{-k_\mathrm{F}-D/v_\mathrm{F}}^{-k_\mathrm{F}+D/v_\mathrm{F}} dk\, v_{k_\mathrm{F},-k_\mathrm{F}|k,-k}\, v_{k,-k|k_\mathrm{F},-k_\mathrm{F}}$$

$$\times \int d\omega\, G^\mathrm{R}_{0,k}\left(\tfrac{\Omega}{2} + \omega\right) G^\mathrm{K}_{0,-k}\left(\tfrac{\Omega}{2} - \omega\right)$$

$$\sim \int_{-D}^{D} d\epsilon\, \frac{1}{\Omega - 2\epsilon + i\eta}\, \mathrm{sgn}(\epsilon)$$

$$= \frac{1}{2} \ln \frac{(\Omega + i\eta)^2}{(\Omega + i\eta)^2 - 4D^2} \overset{|\Omega| \ll D}{\sim} \ln \frac{|\Omega|}{2D}$$

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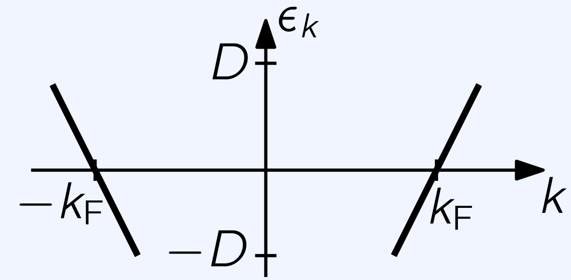
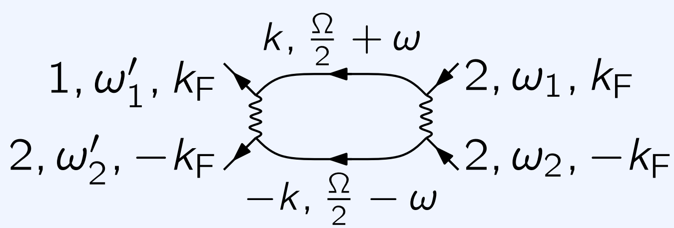




$$\sim \int_{-k_{\mathrm{F}}-D/v_{\mathrm{F}}}^{-k_{\mathrm{F}}+D/v_{\mathrm{F}}} dk\, v_{k_{\mathrm{F}},-k_{\mathrm{F}}|k,-k}\, v_{k,-k|k_{\mathrm{F}},-k_{\mathrm{F}}}$$

$\approx v(2k_{\mathrm{F}})$

$$\times \int d\omega\, G^{\mathrm{R}}_{0,k}\left(\tfrac{\Omega}{2}+\omega\right) G^{\mathrm{K}}_{0,-k}\left(\tfrac{\Omega}{2}-\omega\right)$$

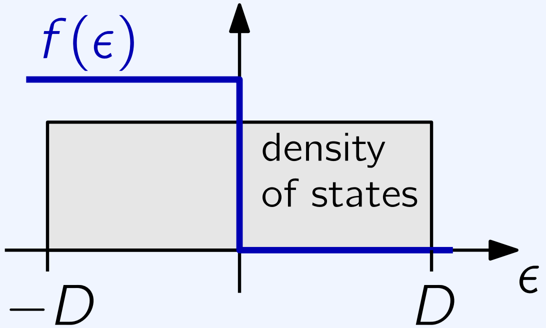
$$G_0^{\mathrm{R}} = \frac{1}{\frac{\Omega}{2} + \omega - \epsilon_k + i\eta}$$

$$G_0^{\mathrm{K}} = -2\pi i \delta\left(\tfrac{\Omega}{2} - \omega - \epsilon_k\right) \times \underbrace{[1 - 2f(\epsilon_k)]}_{\mathrm{sgn}(\epsilon_k)}$$

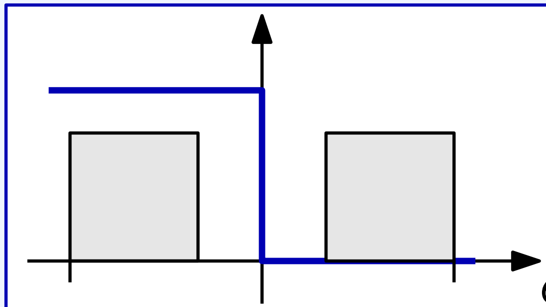
$$\sim \int_{-D}^{D} d\epsilon\, \frac{1}{\Omega - 2\epsilon + i\eta}\, \mathrm{sgn}(\epsilon)$$

$$= \frac{1}{2} \ln \frac{(\Omega + i\eta)^2}{(\Omega + i\eta)^2 - 4D^2} \overset{|\Omega| \ll D}{\sim} \ln \frac{|\Omega|}{2D}$$

Regularizing the divergence

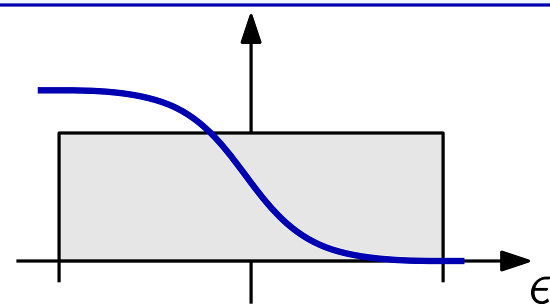


$$\int_{-D}^{D} d\epsilon \, \frac{1}{\Omega - 2\epsilon + i\eta} \, \mathrm{sgn}(\epsilon) \;\sim\; \ln \frac{|\Omega|}{2D}$$



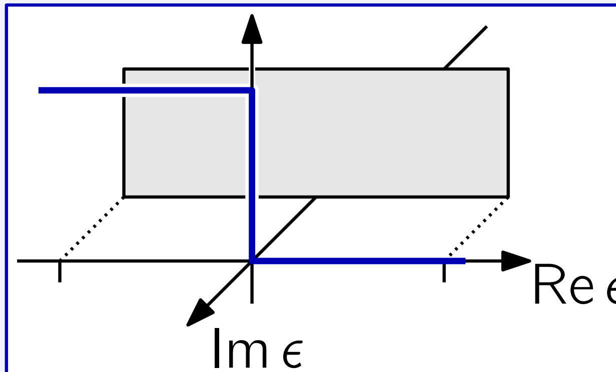
open band gap

- momentum cut-off in bulk systems



smoothen Fermi function

- temperature flow
- imaginary freq. cut-off



enhance decay rate

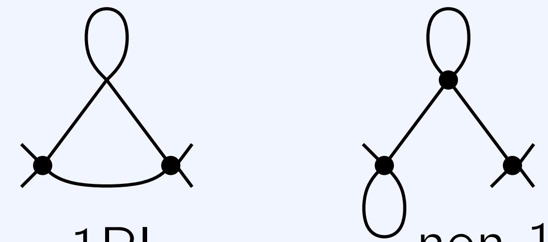
- hybridization flow

$$\left[\int_{-D}^{-\Lambda} + \int_{\Lambda}^{D}\right] d\epsilon \, \frac{1}{\Omega - 2\epsilon + i\Gamma} [1 - 2f_T(\epsilon)] \;\sim\; \ln \frac{\max\{|\Omega|, \Lambda, T, \Gamma\}}{2D}$$

FRG: cut-off in free propagator, $G_0 \to G_{0,\lambda}$ $\to$ flow of 1PI vertex functions?

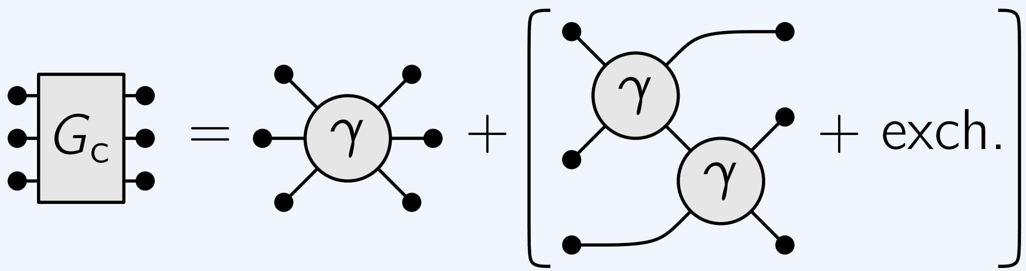
1PI vertex functions

1PI diagram: cannot be disconnected by removing one line



1PI vertex function: sum of all 1PI diagrams (e.g. self-energy)

Reconstruct connected GF by tree diagrams.



Obtain generating functional by Legendre transformation

$$\psi = \frac{\delta\, i \ln \mathcal{G}}{\delta \bar{J}} = \langle a \rangle_{J,\bar{J}} \qquad \bar{\psi} = -\frac{\delta\, i \ln \mathcal{G}}{\delta J} = \langle a^\dagger \rangle_{J,\bar{J}}$$

$$\Gamma[\psi, \bar{\psi}] = \Big(\bar{J} \cdot \psi + \bar{\psi} \cdot J - i \ln \mathcal{G}[J, \bar{J}] \Big)_{\substack{J[\psi,\bar{\psi}] \\ \bar{J}[\psi,\bar{\psi}]}} - \text{(noninteracting result)}$$

$$\gamma(1' \dots n' | 1 \dots n) = (-1)^n \frac{\delta^{2n}}{\delta \bar{\psi}_{1'} \dots \delta \bar{\psi}_{n'} \delta \psi_n \dots \delta \psi_1} \Gamma[\psi, \bar{\psi}] \Big|_{\psi = 0 = \bar{\psi}}$$

Flow equations

$$\mathcal{G}_\lambda = \frac{1}{Z}\int \mathcal{D}[\phi,\bar{\phi}]\,\exp\left[i\bar{\phi}\cdot Q_\lambda\phi + iS_{\text{int}} - i\bar{J}\cdot\phi - i\bar{\phi}\cdot J\right]$$

$$\Gamma_\lambda[\psi,\bar{\psi}] = \Big(\bar{J}\cdot\psi + \bar{\psi}\cdot J - i\ln\mathcal{G}_\lambda[J,\bar{J}]\Big)_{\substack{J_\lambda[\psi,\bar{\psi}]\\ \bar{J}_\lambda[\psi,\bar{\psi}]}} - \text{(noninteracting result)}$$

$$\mathring{\mathcal{G}} = \frac{1}{Z}\int \mathcal{D}[\phi,\bar{\phi}]\; i\bar{\phi}\cdot\mathring{Q}\phi\; e^{\,i\bar{\phi}\cdot Q\phi + \ldots} = -i\,\mathrm{Tr}\,\mathring{Q}\,\frac{\delta^2\mathcal{G}}{\delta\bar{J}\,\delta J}$$

$$\rightarrow \quad \frac{d}{d\lambda}\ln\mathcal{G} \;\rightarrow\; \frac{d}{d\lambda}\ln\mathcal{G}[J_\lambda,\bar{J}_\lambda] \;\rightarrow\; \mathring{\Gamma}[\psi,\bar{\psi}] \;\rightarrow\; \mathring{\gamma}(1'\ldots|\ldots n)$$

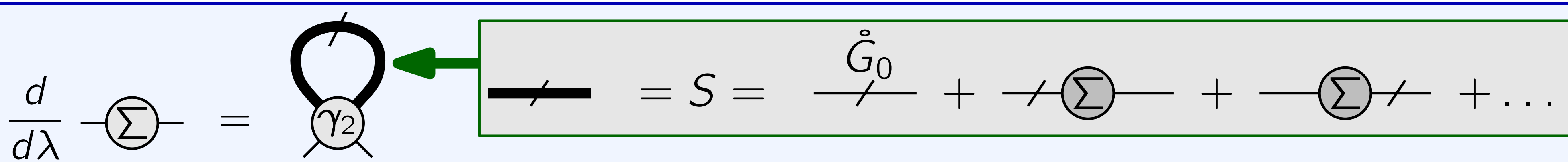
Flow equations

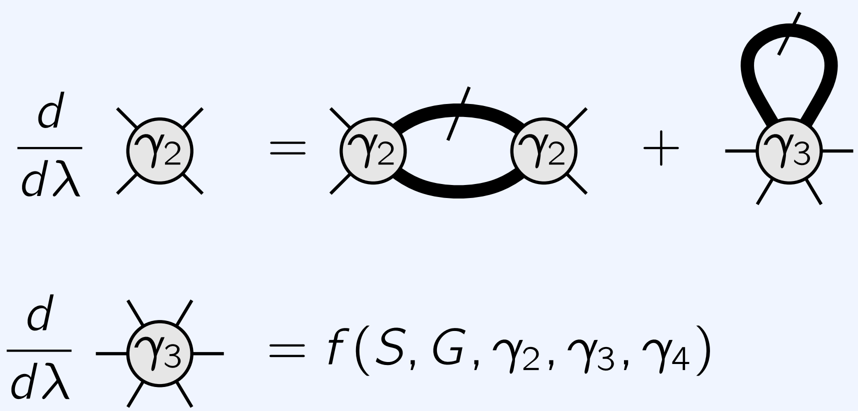
$$\mathcal{G}_\lambda = \frac{1}{Z}\int \mathcal{D}[\phi,\bar\phi]\, \exp\left[i\bar\phi\cdot Q_\lambda\phi + iS_{\text{int}} - i\bar J\cdot\phi - i\bar\phi\cdot J\right]$$

$$\Gamma_\lambda[\psi,\bar\psi] = \Big(\bar J\cdot\psi + \bar\psi\cdot J - i\ln\mathcal{G}_\lambda[J,\bar J]\Big)_{\substack{J_\lambda[\psi,\bar\psi]\\ \bar J_\lambda[\psi,\bar\psi]}} - \text{(noninteracting result)}$$

$$\mathring{\mathcal{G}} = \frac{1}{Z}\int \mathcal{D}[\phi,\bar\phi]\; i\bar\phi\cdot\mathring{Q}\phi\; e^{\,i\bar\phi\cdot Q\phi + \dots} = -i\,\mathrm{Tr}\,\mathring{Q}\frac{\delta^2\mathcal{G}}{\delta\bar J\,\delta J}$$

$$\to \frac{d}{d\lambda}\ln\mathcal{G} \to \frac{d}{d\lambda}\ln\mathcal{G}[J_\lambda,\bar J_\lambda] \to \mathring{\Gamma}[\psi,\bar\psi] \to \mathring{\gamma}(1'\dots|\dots n)$$





Infinite coupled hierarchy.

Approximations:

- Neglect $\mathring{\gamma}_n$ for $n \geq n_0$ ("truncation")
- Parameterize vertex functions

Flow equations

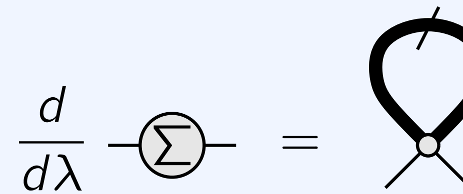
Lowest order truncation:

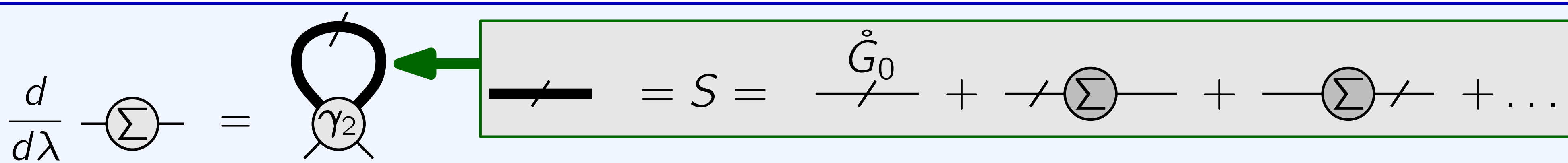
- static self-energy ($\rightarrow$ shift of levels)
- often captures exponents in leading order

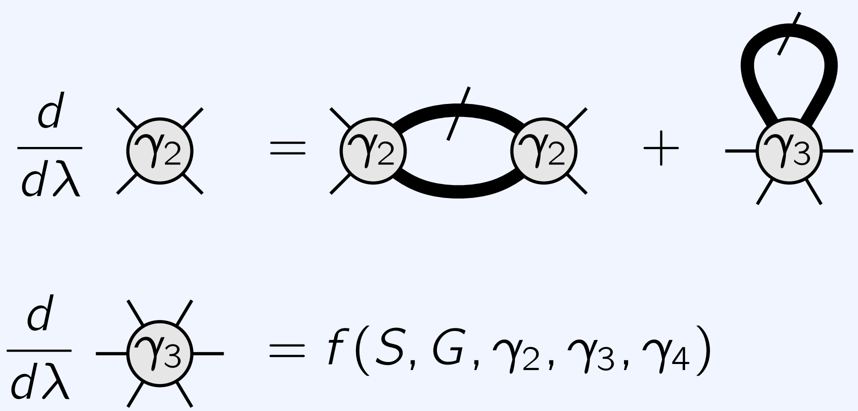
For interaction induced dephasing:

higher order truncation

$\rightarrow$ numerically more demanding







Infinite coupled hierarchy.

Approximations:

- Neglect $\mathring{\gamma}_n$ for $n \geq n_0$ ("truncation")
- Parameterize vertex functions

Overview

1. Keldysh formalism
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2. **Functional RG** →
3. Cut-offs for the Keldysh FRG
4. Examples:
   - Wire with strong contact barriers in steady state noneq.
   - transient behaviour of the IRLM after attaching leads

- Divergencies regularized by:
  band gap, temperature, decay rate
- Flow of 1PIs: infinite coupled hierarchy
- Lowest order truncation
  → level renormalization

Review:

Metzner et al., Rev. Mod. Phys. **84**, 299 (2012)

Book:

Kopietz, Bartosch, Schütz, Springer 2010

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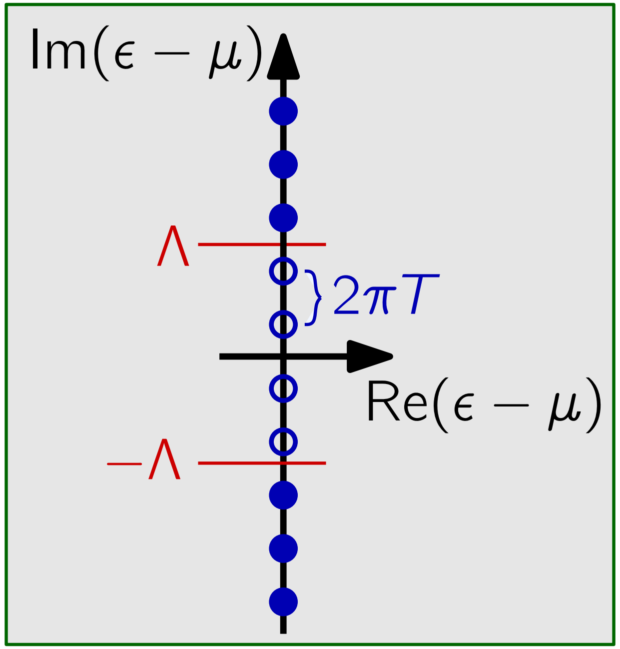
Imaginary frequency cut-off in Keldysh-fRG

Cut-off for Matsubara-FRG:

$$G^{\text{eq}}_{0,\Lambda}(i\omega_n) = \Theta(|\omega_n| - \Lambda)\frac{e^{i\omega_n \eta}}{i\omega_n - \epsilon + \mu}$$



Keldysh counter-part:

$$f(\epsilon_k) = \langle a^\dagger_k a_k \rangle_0 = G^{\text{eq}}_{0,k}(\tau = 0)$$

$$= \frac{1}{\beta}\sum_{\omega_n} G^{\text{eq}}_{0,k}(i\omega_n)$$

$$f_\Lambda(\epsilon) = T \sum_{\omega_n} \frac{\Theta(|\omega_n| - \Lambda)e^{i\omega_n \eta}}{i\omega_n - \epsilon + \mu}$$

Poles close to real axis removed
$\rightarrow$ step of Fermi function smoothened

in particular

$$f_\Lambda(\epsilon) = \tfrac{1}{2} \text{ for } T, \mu, \epsilon \ll \Lambda \ll \tfrac{1}{\eta}$$

corresponds to $T = \infty$

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corresponds to $T = \infty$

$$\Sigma^{\text{res,K}}_\Lambda(\omega) = -2i \sum_{r=\text{L,R}} [1 - 2f_{r,\Lambda}(\omega)]\Gamma_r(\omega)$$

Reservoir dressed noninteracting dot propagator:

$$G^{\text{K}}_{0,\Lambda}(\omega) = G^{\text{R}}_0(\omega)\Sigma^{\text{res,K}}_\Lambda(\omega)G^{\text{A}}_0(\omega)$$

Properties:

- Cutting sum is technically convenient
- Equilibrium, static truncation: identical to Matsubara fRG
- Higher truncation:
  - violation of equilibrium relations
  - not identical to Matsubara fRG

Imaginary frequency cut-off in Keldysh-fRG

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Equilibrium relations in Keldysh formalism:

- single-particle: FDT

  $$\Sigma^{\mathrm{K}}(\omega) = [1 - 2f(\omega)][\Sigma^{\mathrm{R}}(\omega) - \Sigma^{\mathrm{A}}(\omega)]$$

- multi-particle: KMS-relations (more complicated)

Hybridization flow

Fermi liquid relation for Anderson impurity in equilibrium:

$$\Sigma^{R}_{\sigma}(\omega, T) = \frac{U}{2} + \left(1 - \frac{\chi_{s} + \chi_{c}}{2}\right)(\omega - \mu) - i\,\frac{(\chi_{s} - \chi_{c})^2}{2\Gamma}\left[(\omega-\mu)^2 + (\pi T)^2\right] + \ldots$$

- influences shape of Kondo resonance
- based on equilibrium particle statistic
- destroyed by truncated imaginary-freq. flow

Flow parameter:

$$G^{R}_{0,\Lambda}(t|t') = e^{-\Lambda(t-t')} G^{R}_{0}(t|t')$$

$$G^{R}_{0,\Lambda}(\omega) = \frac{1}{\omega - \epsilon + i(\Gamma + \Lambda)}$$

$$\Lambda : \infty \to 0$$

Couple each dot state to auxiliary reservoir with hybridization Λ.

Properties:

- Eq. relations preserved.
  Advantage: physical meaning
- Not a "cut-off"
  $\rightarrow$ flow numerically more demanding
- Equilibrium, static truncation, $T = 0$: identical to Matsubara fRG
- freedom to choose T and μ of auxiliary reservoirs

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- numerically less efficient

Keldysh cut-offs: SJ, thesis, Aachen 2010

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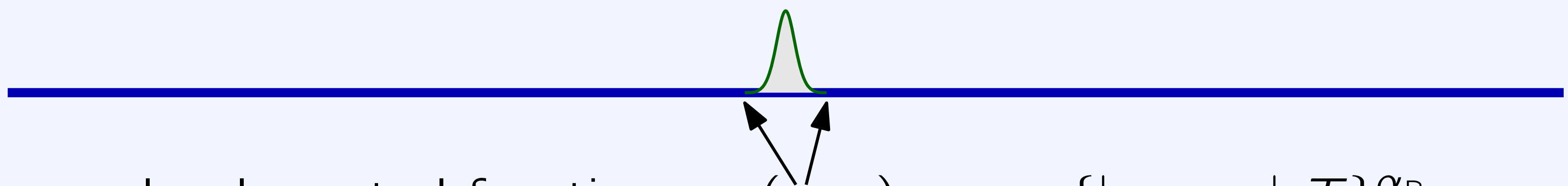
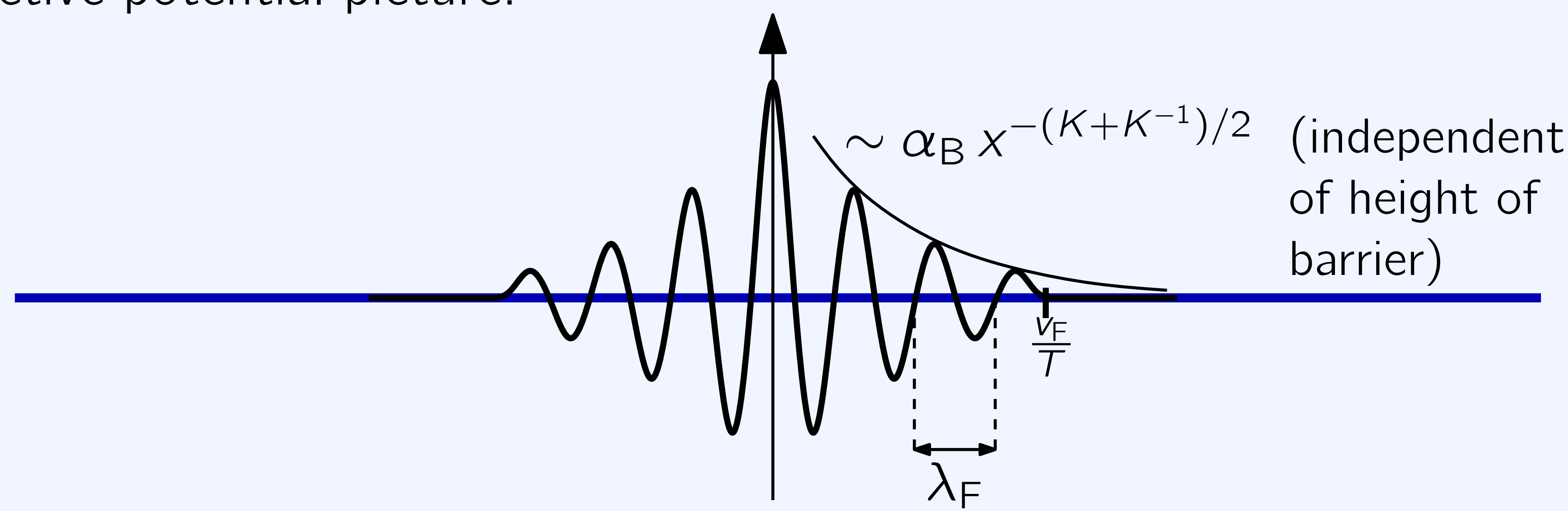
Luttinger liquid with barrier



local spectral function: $\rho(x,\omega) \sim \max\{|\omega - \epsilon_{\mathrm{F}}|, T\}^{\alpha_{\mathrm{B}}}$

$$\alpha_{\mathrm{B}} = \text{boundary exp.} = \frac{1}{K} - 1, \qquad K = \text{"stiffness"} \begin{cases} = 1, & \text{no interact.} \\ < 1, & \text{repulsive int.} \end{cases}$$

Effective potential picture:

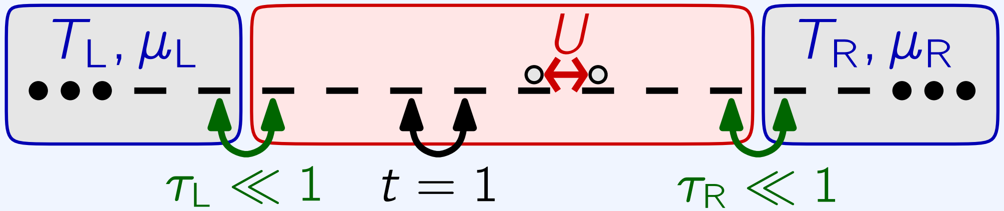


transmission: $|t(\omega)|^2 \sim \max\{|\omega - \epsilon_{\mathrm{F}}|, T\}^{2\alpha_{\mathrm{B}}}$

conductance: $G(T) = \int d\omega \left(-\frac{\partial f}{\partial \omega}\right) |t(\omega)|^2 \sim T^{2\alpha_{\mathrm{B}}}$

Luttinger liquid with two contact barriers

The model



spinless fermions

$$H_{\text{wire}} = -\sum_{i=1}^{N} c_{i+1}^{\dagger} c_i + \text{Hc}, \quad N \sim 10^4$$

$$H_r = -\sum_{i=1}^{\infty} a_{r,i+1}^{\dagger} a_{r,i} + \text{Hc}, \quad r = \text{L}, \text{R}$$

$$H_{\text{tun}} = -\tau_{\text{L}} a_{\text{L},1}^{\dagger} c_1 - \tau_{\text{R}} a_{\text{R},1}^{\dagger} c_N$$

$$\tau_{\text{L,R}} \ll 1$$

$$H_{\text{int}} = U \sum_{i=1}^{N-1} \left(n_i - \tfrac{1}{2}\right)\left(n_{i+1} - \tfrac{1}{2}\right)$$

Nonequilibrium fRG

- Imaginary frequency cut-off
- Static truncation: $\gamma_2^{\Lambda} = \overline{v}$

  $\Rightarrow$ effective potential
- in equilibrium:

  exponent = correct $+ \mathcal{O}(U^2)$
- lowest order in coupling $\Gamma \ll 1$:

$$f(\epsilon) = \frac{\Gamma_{\text{L}} f_{\text{L}}(\epsilon) + \Gamma_{\text{R}} f_{\text{R}}(\epsilon)}{\Gamma}$$

$$\Rightarrow \partial_\Lambda \Sigma_\Lambda^{\text{Ret}} = \sum_{r=\text{R,L}} \frac{\Gamma_r}{\Gamma} \left(\partial_\Lambda \Sigma_\Lambda^{\text{eq}}\right)_{\mu=\mu_r}$$

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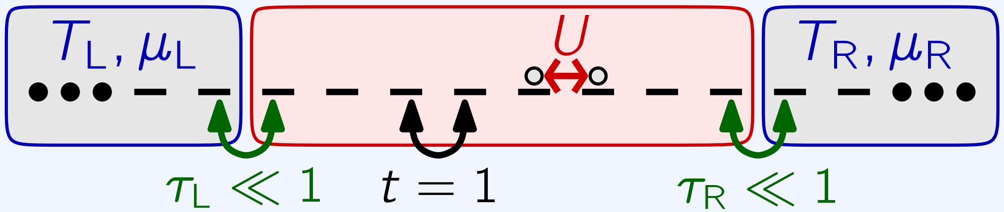
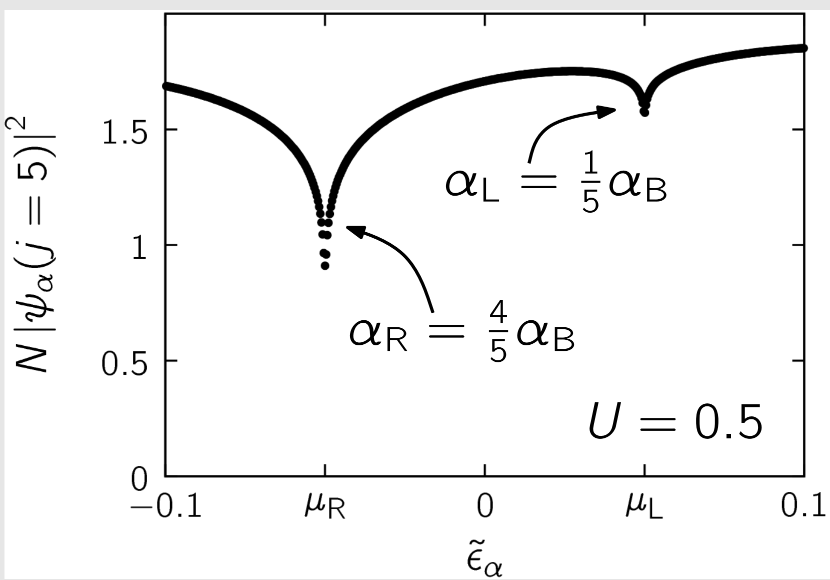


spinless fermions

$$\rho(\omega) \sim |\omega - \mu_\mathrm{L}|^{\alpha_\mathrm{L}} \cdot |\omega - \mu_\mathrm{R}|^{\alpha_\mathrm{R}}$$

$$\alpha_r = \frac{\Gamma_r}{\Gamma} \alpha_\mathrm{B}(\mu_r)$$



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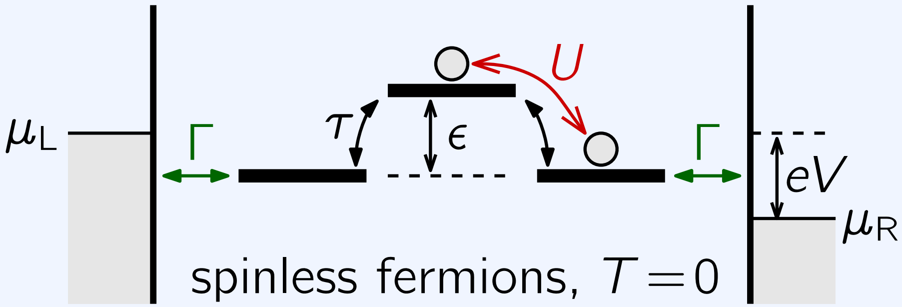
Two oscillating potentials superimposed

- wavelength $= \lambda(\mu_r)$
- amplitude $\sim \Gamma_r$

IRLM in the transient regime



$$H_{\text{dot}} = \epsilon a_2^\dagger a_2 + \tau(a_1^\dagger a_2 + a_2^\dagger a_3 + \text{H.c.})$$

$$H_{\text{int}} = U\left[\left(n_1 - \tfrac{1}{2}\right)\left(n_2 - \tfrac{1}{2}\right) + \left(n_2 - \tfrac{1}{2}\right)\left(n_3 - \tfrac{1}{2}\right)\right]$$

Reservoirs in wide band limit:

$$\Sigma_r^{\text{R}}(t'|t) = -i\delta(t'-t)\Gamma_r$$

$$\Sigma_r^{\text{K}}(t'|t) = -\frac{2}{\pi}\frac{e^{-i\mu_r(t'-t)}}{t'-t}\Gamma_r$$

- Equilibrium: First order PT fails:

$$\frac{\tau_{\text{pert}}}{\tau} = 1 - \frac{2U}{\pi\Gamma}\ln\frac{\tau}{\Gamma}$$

FRG yields

$$\frac{\tau_{\text{ren}}}{\tau} \sim \left(\frac{\tau}{\Gamma}\right)^{-2U/(\pi\Gamma)+\mathcal{O}(U^2)}$$

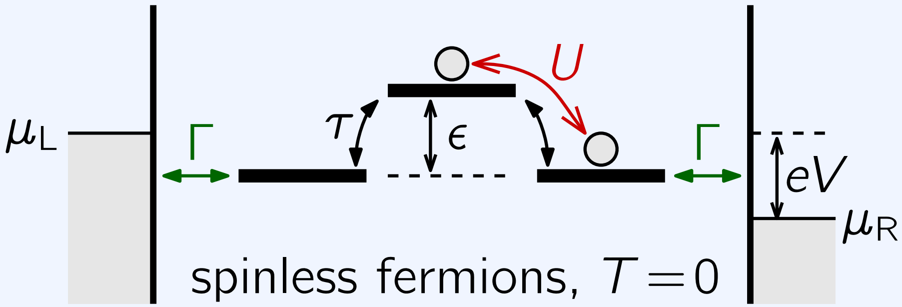
- Scaling limit: $\tau, \epsilon, U, V \ll \Gamma$
  $\rightarrow$ equivalence to single-site model

- Low-energy physics governed by T_K,

$$\chi = \left.\frac{d\langle n_2\rangle}{d\epsilon}\right|_{\epsilon=0} = -\frac{2}{\pi T_K}$$

$$T_K(U=0) = 4\tau^2/\Gamma$$

IRLM in the transient regime



- $t=0$: attach leads to empty dot
- Study occupancy of central site

For $U=0$, $t \gg 1/T_K$, single-site model

$$\bar{n}_2(t) = \bar{n}_{\text{stat}}(1+e^{-T_K t}) + \frac{T_K}{2\pi}\frac{e^{-T_K t/2}}{t} \times \left(\frac{\sin\left[\left(\epsilon - \frac{V}{2}\right)t\right]}{\left(\epsilon - \frac{V}{2}\right)^2} + \frac{\sin\left[\left(\epsilon + \frac{V}{2}\right)t\right]}{\left(\epsilon + \frac{V}{2}\right)^2} \right)$$

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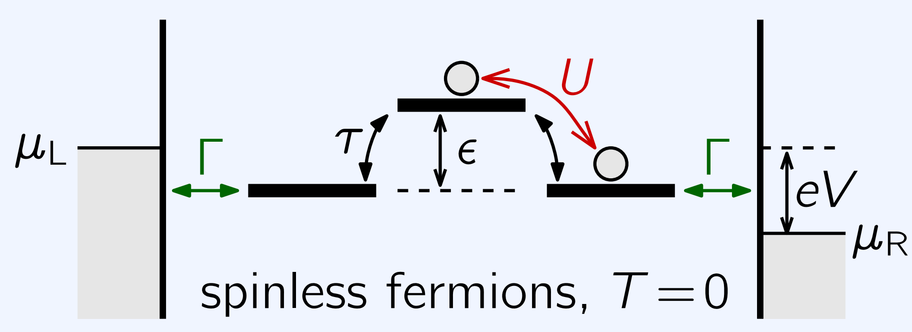

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exponential decay

algebraic decay

oscillations

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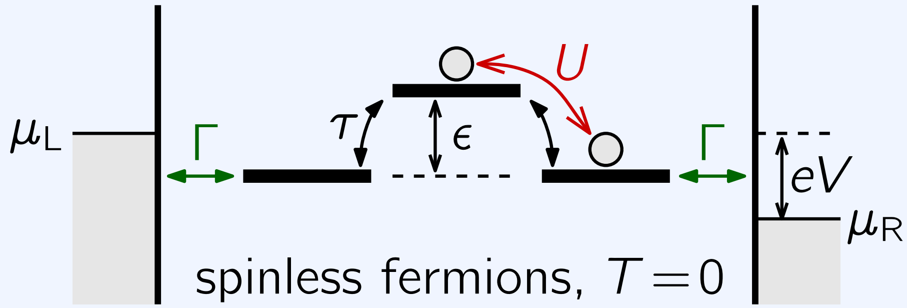

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For $U = 0, t \gg 1/T_\mathrm{K}$, single-site model

exponential decay

$$\overline{n}_2(t) = \overline{n}_\mathrm{stat}(1 + e^{-T_\mathrm{K} t})$$

$$+ \frac{T_\mathrm{K}}{2\pi} \frac{e^{-T_\mathrm{K} t/2}}{t}$$

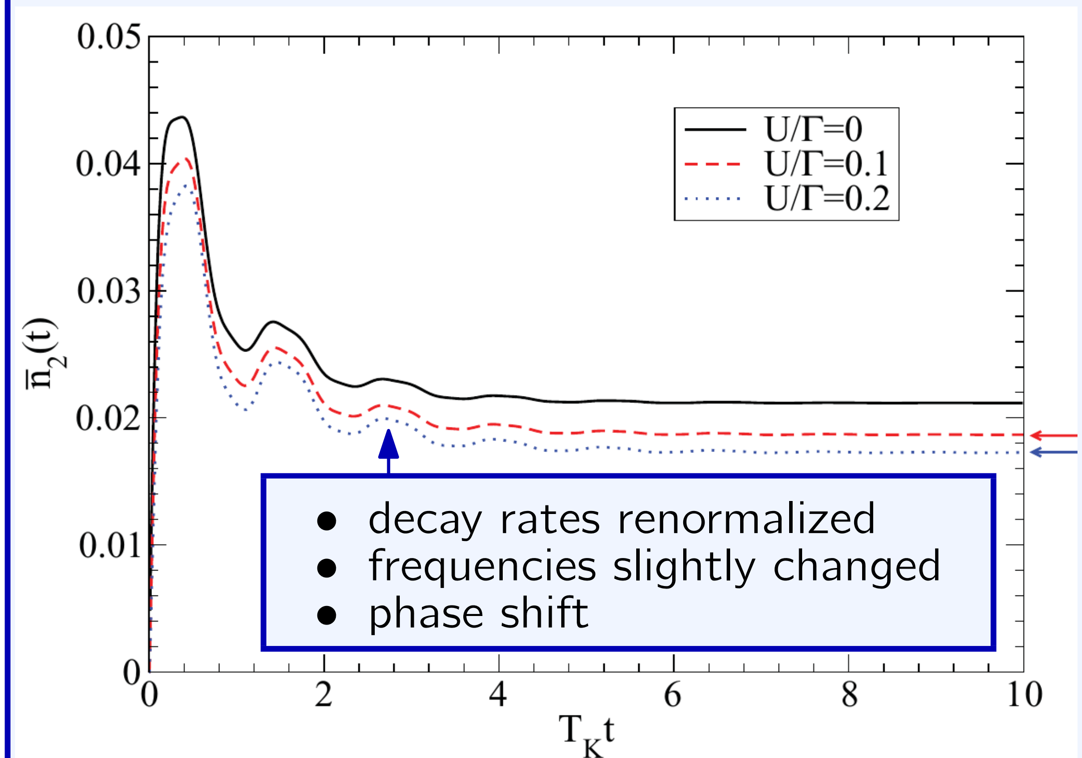
algebraic decay

$$\times \left(\frac{\sin[(\epsilon - \frac{V}{2})t]}{(\epsilon - \frac{V}{2})^2} + \frac{\sin[(\epsilon + \frac{V}{2})t]}{(\epsilon + \frac{V}{2})^2} \right)$$

oscillations

Nonequilibrium fRG

- Hybridization flow
- lowest order truncation



$\tau = \Gamma/40, \epsilon = 10 T_\mathrm{K}, V = 10 T_\mathrm{K}$

- confirmed by comparison to RTRG

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FRG in lowest order truncation:

- fast and intuitive approach to non-perturbative problems
- steady state noneq. and time dependence

Wire with contact barriers:
SJ et al., PRL **99** 150603 (2007)

Transient behaviour of IRLM:
Kennes et al., PRB **85**, 085113 (2012)

Outlook

Some recent applications of Keldysh FRG to mesoscopic systems:

- Quench dynamics of Spin-Boson-model: Kashuba et al., PRB 2013
- Thermotransport of IRLM: Kennes et al., EPL 2013
- Quench dynamics of Luttinger liquid: Kennes, Meden, PRB 2014
- Steady state noneq. of Anderson-Holstein model: Laakso et al., NJP 2014
- Josephson Q-dot with V or T bias: Rentrop et al., PRB 2014

Method development:

Periodically driven systems

$\rightarrow$ Keldysh-fRG + Floquet

talk Katharina Eissing, TT102.6 (Thu, 16:15)

In 1PI-FRG: Higher order truncations numerically very costly

$\rightarrow$ more efficients schemes?

$\rightarrow$ 2PI-FRG?

talk Jan Rentrop, TT86.7 (Thu, 11:00)